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A non-standard quantum double as the l-state boson algebra and its R-matrices for the quantum Yang–Baxter equation

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Abstract. In this paper we construct a new quantum double by endowing the l-state boson algebra with a non-trivial Hopf algebra structure, which is not a q-deformation of the Lie algebra or superalgebra. The universal R-matrix for the Yang–Baxter equation associated with this new quantum group structure is obtained explicitly. By building the representations of this quantum double, we obtain some R-matrices which can result in new representations of the braid group.

1. Introduction

Recently, the quantum Yang–Baxter equation (QYBE) and its quantum group theory have attracted great interest from both theoretical physicists and mathematicians [1–3]. This is because the QYBE is a key to the complete integrability of many physical systems appearing in quantum inverse scattering methods [4, 5], exactly-solvable models in statistical mechanics [6] and low-dimensional quantum field theory [7, 8]. In solving the QYBE and classifying its solutions (R-matrices) in a generally algebraic way, an important mathematical structure—the quasi-triangular Hopf algebra (loosely called the quantum group) has been found in connection with the QYBE [9–12]. Among these studies, the Drinfeld's quantum double (QD) [9] theory provides one with a powerful method to systematically obtain solutions of the QYBE. Some developments and detailed constructions based on Drinfeld's theory have been given by many authors [13–16]. Concrete quantum doubles usually are the 'q-deformations' of certain algebras as non-co-commutative quasi-triangular Hopf algebras and we call them standard quantum doubles. Recent studies show that not only the standard R-matrices [13–15] but also the non-standard ones [17–19], which are obtained by direct matrix calculations, such as the coloured R-matrices [20], can be obtained in the framework of Drinfeld's QD theory. For the latter, the cyclic representations and other non-generic representations at roots of unity [21–28] must be considered [29–32]. In fact, the cyclic representations were also used to construct the R-matrices with non-additive spectrum parameters [33–41]. These studies implied the fact that the 'new' representations of the original standard

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quantum doubles may lead to 'new' R-matrices for QYBE. A subsequent question naturally is whether there exist 'new' quantum doubles that result in 'new' R-matrices. The answer is affirmative. More recently, it has been shown that the parametrization of the standard quantum doubles can also enjoy some non-standard and coloured R-matrices [42, 43] directly. The purpose of the present paper is to search for a class of new quantum doubles that are neither those standard quantum doubles nor their parametrizations and then to use them to find the new universal R-matrices for the QYBE. Since the obtained new quantum doubles are not the q-deformations of any algebras and homomorphisms to the l-state boson algebras ($l=1, 2\ldots$) as associative algebras, we will call them non-standard quantum doubles associated with the boson algebra.

To begin our discussion conveniently, we need to outline some basic ideas in Drinfeld's QD theory (for reviews see [44, 45]) so that the notation used in this paper can be clarified. Suppose we are given two Hopf algebras A, B and a non-degenerate bilinear form $\langle , \rangle : A\times B\rightarrow C$ (the complex field) satisfying the following conditions:

$$\begin{aligned}
&\langle a, b_1b_2\rangle = \langle \Delta_A(a), b_1\otimes b_2\rangle \qquad a\in A, b_1, b_2\in B,\\
&\langle a_1a_2, b\rangle = \langle a_2\otimes a_1, \Delta_B(b)\rangle \qquad a_1, a_2\in A, b\in A\\
&\langle 1_A, b\rangle = \varepsilon_B(b) \qquad b\in B,\\
&\langle a, 1_B\rangle = \varepsilon_A(a) \qquad a\in A\\
&\langle S_A(a), S_B(b)\rangle = \langle a, b\rangle \qquad a\in A, b\in B
\end{aligned} \tag{1.1}$$

where for $C=A, B, \Delta_C$, ε_C and S_C are the co-product, co-unit and antipode of C respectively; 1_C is the unit of C. Drinfeld's QD theory states the central results as follows.

Using (1.1), we can find a Hopf algebra D, the quantum double, satisfying the following conditions

1. D contains A and B as Hopf subalgebras;
2. The mapping $A\times B\rightarrow D: a\otimes b\rightarrow ab$ is an isomorphism of vector space;
3. For any $a\in A, b\in B$, we have the multiplication

$$ba = \sum_{i,j} \langle a_i(1), S(b_j(1))\rangle \langle a_i(3), b_j(3)\rangle a_i(2)b_j(2) \tag{1.2}$$

where $c_i(k)(k=1, 2, 3; c=a, b)$ are defined by

$$\Delta^2(c) = (id\otimes\Delta)\Delta(c) = (\Delta\otimes id)\Delta(c) = \sum_i c_i(1)\otimes c_i(2)\otimes c_i(3).$$

Furthermore there exists an unique element

$$\hat{R} = \sum_m a_m\otimes b_m \in A\times B\subset D\times D$$

obeying the 'abstract' QYBE

$$\hat{R}_{12}\hat{R}_{13}\hat{R}_{23} = \hat{R}_{23}\hat{R}_{13}\hat{R}_{12} \tag{1.3}$$

where a_m and b_m are the basis vectors of A and B respectively, and they are dual to each other, i.e. $\langle a_m, b_n\rangle = \delta_{m,n}$;

$$\hat{R}_{12} = \sum_m a_m\otimes b_m\otimes 1, \ \hat{R}_{13} = \sum_m a_m\otimes 1\otimes b_m, \ \hat{R}_{23} = \sum_m 1\otimes a_m\otimes b_m.$$

Notice that the usual quantum double is obtained by taking the subalgebra A in the above construction to be a Borel subalgebra of the universal enveloping algebra

(UEA) of the classical Lie algebra and Lie superalgebra [14]. It is only a q-deformation of the UEA or its parametrization. In this paper we will build a different quantum double based on Drinfeld's theory. This paper is arranged as follows. In section 2, we take a 'half' of the l-state boson algebra as the Hopf subalgebra A with co-commutative coproduct in the QD construction and then built its quantum dual as a non-co-commutative but commutative Hopf subalgebra B. Then, we combine A and B to form the so-called non-standard quantum double D and thereby obtain the new universal R-matrix for the QYBE. In section 3, we generally study the representation theory of this quantum double and built the basic construction for its representations. In section 4, we use explicit finite dimensional representations to obtain the new R-matrices for the QYBE from the obtained universal R-matrix. Finally, in section 5, we give some remarks on the construction in this paper and its relations to some recent works given by both other and present authors.

2. The non-standard quantum double D as a boson algebra and its universal R-matrix

Let us consider an associative (C) algebra A generated by $\bar{N}, \bar{a}_1, \bar{a}_2, \ldots, \bar{a}_l$ satisfying

$$[\bar{N}, \bar{a}_i] = \bar{a}_i, \ [\bar{a}_i, \bar{a}_j] = 0. \tag{2.1}$$

We can regard A as a 'half' of the l-state boson algebra generated by creation operators $a_i^+ = \bar{a}_i$, annihilation operators a_i and the total number operator $\bar{N}$. By endowing A with the following structures (Δ, ε, S):

$$\begin{aligned} &\Delta: A \to A \times A: \Delta(xy) = \Delta(x)\Delta(y) && \Delta(x) = x \otimes 1 + 1 \otimes x \\ &\varepsilon: A \to C: \ \varepsilon(xy) = \varepsilon(x)\varepsilon(y) && \varepsilon(x) = 0, \ \varepsilon(1) = 1 \\ &S: A \to A: S(xy) = S(y)S(x) && S(x) = -S(x), \ S(1) = 1. \end{aligned} \tag{2.2}$$

For $x = \bar{N}, \bar{a}_i$, the algebra A becomes a co-commutative Hopf algebra. In fact, it is trivial to define such a Hopf algebra since we can regard A as a universal enveloping algebra of a Lie algebra with basis $\bar{N}, \bar{a}_i (i = 1, 2, \ldots)$. However, since A is non-commutative, its dual $A^* = B$ with opposite co-product $\Delta_B \equiv \Delta$ must be non-co-commutative. Now, we try to determine the generators and the structure relation for B.

According to the Poincaré–Birkhoff–Witt (PBW) theorem, the basis for A can be chosen as

$$\bar{e}(m, n_i) = \bar{e}(m; n_1, n_2, \ldots, n_l) = \bar{a}_1^{n_1} \bar{a}_2^{n_2} \ldots \bar{a}_l^{n_l} \bar{N}^m$$

where $m, n_1, n_2, \ldots, n_l \in Z^+ = \{0, 1, 2, \ldots\}$. On this basis, the generators N and $a_i (i = 1, 2, \ldots, l)$ dual to $\bar{N}$ and $\bar{a}_i$ respectively are defined by the following conditions

$$\begin{aligned} &\langle \bar{N}, N \rangle = 1 && \langle x, N \rangle = 0 \\ &\langle \bar{a}_i, a_i \rangle = 1 && \langle y, a_i \rangle = 0 \end{aligned} \tag{2.3}$$

where x is a basis element other than $\bar{N}$ and y other than $\bar{a}_i$.

Since the dual of a co-commutative Hopf algebra must be commutative, we immediately have

Proposition 1. B is an Abelian algebra with the commutative generators N and a_i ($i = 1, 2, \ldots, l$).

Let us extend the bilinear mapping $\langle , \rangle$ defined only for the generators to all the vector pairs in $A \times B$.

Proposition 2.

$$\begin{aligned}&\langle \bar{N}^m, N^n\rangle = \delta_{m,n} \qquad \langle \bar{a}_i^m, a_j^n\rangle = m!\delta_{m,n}\delta_{i,j}\\&\langle \bar{e}(m, n_i), a_1^{r_1}a_2^{r_2}\ldots a_l^{r_l}N^s\rangle = m!n_1!n_2!\ldots n_l!\delta_{m,s}\delta_{n_1,r_1}\ldots\delta_{n_l,r_l}.\end{aligned} \tag{2.4}$$

Proof.

$$\begin{aligned}\langle \bar{N}^m, N^n\rangle &= \langle \Delta(\bar{N}^m) \qquad N^{n-1}\otimes N\rangle = \langle \Delta(\bar{N})^m, N^{n-1}\otimes N\rangle\\&= \sum_{i=0}^{m}\frac{m!}{(m-i)!i!}\langle \bar{N}^i, N^{n-1}\rangle\langle \bar{N}^{m-i}, N\rangle\\&= m\langle \bar{N}^{m-1}, N^{n-1}\rangle = \ldots = m!\delta_{m,n}.\end{aligned}$$

The other parts of this proposition can be proved in a similar way.

As for the Hopf algebra structure of B, we have to find the coproduct Δ, the antipode S and the co-unit ε.

Proposition 3.

$$\begin{aligned}&\Delta(N) = N\otimes 1 + 1\otimes N,\ \Delta(a_i) = a_i\otimes e^N + 1\otimes a_i,\ i = 1, 2, \ldots, l)\\&S(N) = -N,\ S(a_i) = -e^{-N}a_i,\ S(1) = 1\\&\varepsilon(N) = 0 = \varepsilon(a_i) \qquad \varepsilon(1) = 1.\end{aligned} \tag{2.5}$$

Proof. Notice that the linear form $\langle , \Delta(a_i)\rangle$ is non-zero only on $\bar{a}_i\otimes 1$, $1\otimes\bar{a}_i$, $\bar{a}_i\otimes\bar{N}^m (m = 1, 2, \ldots)$. In fact,

$$\begin{aligned}&\langle \bar{a}_i\otimes 1, \Delta(a_i)\rangle = \langle 1\otimes\bar{a}_i, \Delta(a_i)\rangle = \langle \bar{a}_i, a_i\rangle = 1\\&\langle \bar{a}_i\otimes\bar{N}^m, \Delta(a_i)\rangle = \langle \bar{N}^m\bar{a}_i, a_i\rangle = \langle \bar{a}_i(\bar{N}+1)^m, a_i\rangle = 1.\end{aligned}$$

Consequently, we have

$$\begin{aligned}\Delta(a_i) &= a_i\otimes\sum_{k=0}^{\infty}\frac{N^k}{k!} + 1\otimes a_i\\&= a_i\otimes e^N + 1\otimes a_i.\end{aligned}$$

The other parts of this proposition can be proved in the same way.

Knowing the Hopf algebraic structure of B, we need to derive the relations between A and B so that they are combined with each other to form a new quantum double D.

Proposition 4.

$$\begin{aligned}&[\bar{a}_i, a_j] = \delta_{i,j}(e^N - 1)\\&[N, a_i) = -a_i\\&[N, \text{everything}] = 0.\end{aligned} \tag{2.6}$$

Proof. Using the (1.2) and

$$\Delta^2(a_i)=a_i\otimes e^N\otimes e^N+1\otimes a_i\otimes e^N+1\otimes 1\otimes a_i$$

$$\Delta^2(\bar{a}_i)=\bar{a}_i\otimes 1\otimes 1+1\otimes\bar{a}_i\otimes 1+1\otimes_l\otimes\bar{a}_i$$

we have

$$a_i\bar{a}_i=\langle\bar{a}_i,\,S(a_i)\rangle\langle 1.e^N\rangle 1.e^N+\langle 1.1\rangle\langle 1.e^N\rangle\bar{a}_i a_i+\langle 1,\,1\rangle\langle\bar{a}_i,\,a_i\rangle 1.1$$
$$=-e^N+\bar{a}_i a_i+1$$

which lead to the first question in (2.6). The other equations can be proved in the same way.

From propositions 1–4, we know that the basis for A dual to $\bar{e}[m, n_i]$ can be chosen as

$$e[m.n_i]=\frac{a_1^{n_1}a_2^{n_2}\ldots a_l^{n_l}N^m}{m!n_1!n_2!\ldots n_l!}. \tag{2.8}$$

Then, we immediately write down the universal R-matrix of D

$$\hat{R}=\sum_{[m,n_i]}\bar{e}[m,\,n_i]\otimes e[m,\,n_i]$$
$$=\prod_{i=1}^{l}\exp(\bar{a}_i\otimes a_i)\cdot\exp(\bar{N}\otimes N). \tag{2.9}$$

3. On the representation theory for the quantum double D

To get finite dimensional R-matrices from the universal R-matrix, we have to find the non-trivial finite dimensional representations of the quantum double D. In the following discussion, a representation T of D in which $T(x)=0$ for certain generators x of D are thought to be trivial.

Proposition 5. All the non-trivial irreducible representation of D must be infinite dimensional.

Proof. We prove this proposition only for the case of $l=2$ and the proof for the arbitrary l is routine. Suppose that there exists a non-trivial finite dimensional irreducible representation $T:D\to\mathrm{End}(V)$ (for simplicity we denote $T(x)$ by x in the following). According to the Schur lemma, the central element N must be a non-zero scalar ξ, i.e. $N=\xi\neq 0$. For the algebraically-closed field C, there exists a vector v such that

$$\bar{N}v=\eta v(\eta\in C).$$

Since a series of eigenvectors $v,\bar{a}_1,\bar{a}_1^2v,\ldots,\bar{a}_1^nv,\ldots,$ of N correspond to different eigenvalues $\eta,\eta+1,\ldots,\eta+n,\ldots,$ they are independent linearly. Due to the finite dimension of V there must be a non-zero extreme vector u such that

$$u=\bar{a}_1^lv,\ \bar{a}_1u=0.$$

Similarly, for other vector series $u, \bar{a}_2 u, \bar{a}_2^2 u, \ldots$, we have

$$\bar{a}_2(\bar{a}_2^{s-1})u = \bar{a}_2^s u = 0.$$

Let $u(2) = \bar{a}_2^{s-1}u$. Then,

$$\bar{a}_i u(2) = 0 \qquad i = 1, 2.$$

It can be proved that

$$W = \mathrm{span}\{f(m, n) = a_1^m a_2^n u(2) | m, n \in Z^+\}$$

is an invariant subspace in V under the action of D. Because V is finite dimensional, there must be w and $z \in Z^+$ such that

$$\begin{aligned} a_1 f(w-1, 0) &= a_1^w u(2) = 0 \\ a_2 f(0, z-1) &= a_2^z u(2) = 0 \end{aligned} \tag{3.1}$$

that is to say, the dimension of W is zw. Because of the irreducibility of V, $W = V$. On other hand,

$$0 = \bar{a}_1 a_1^w u(2) = -w(1 - e^{\xi}) a_1^{w-1} u(2)$$

that is $w = 0$; similarly $z = 0$. Then, one comes to a contradictory conclusion that the representation space V has dimension zero!

According to the above proposition, the non-trivial finite dimensional representations of D are only indecomposable, i.e. reducible but not completely reducible, if they are not the direct sums of some trivial representations. Therefore, the non-trivial R-matrices of D should be associated with the indecomposable representations of D.

To construct such representations of D explicitly, let us define the Fock-like space $F(l)$:

$$F(l) = \mathrm{span}\{|m_i, p\rangle = \bar{a}_1^{m_1} \bar{a}_2^{m_2} \ldots \bar{a}_l^{m_l} E^p |0\rangle | p, m_i = 0, 1, 2, \ldots; i = 1, 2, \ldots, l\}$$

where the vacuum-like state $|0\rangle$ obeys

$$a_i|0\rangle = o, \bar{N}|0\rangle = \mu|0\rangle \qquad \mu \in C, i = 1, 2, \ldots l.$$

On this space, we get an infinite dimensional representation $\rho(\mu)$:

$$\begin{aligned} \bar{a}_i|m_j, p\rangle &= |m_j + \delta_{i,j}, P\rangle \\ a_i|m_j, p\rangle &= m_i|m_j - \delta_{i,j}, p+1\rangle \\ E|m-j, p\rangle &= |m_j, p+1\rangle \\ \bar{N}|m_j, p\rangle &= (m_1 + m_2 + \ldots + m_l + \mu)|m_j, p\rangle \end{aligned} \tag{3.2}$$

where we have defined

$$N = \ln(1 - E) \qquad \text{or} \qquad E = 1 - e^N.$$

Note that all the vectors $|m_j, p\rangle$ satisfying $m_1 + m_2 + \ldots + m_l + p \geqslant K \in Z^+$ span an invariant subspace $V(K)$. Its quotient space

$$Q(K, \mu) = F(l)/V(K) = \mathrm{span}\{|m^j, p\rangle \bmod V(K) | m_1 + m_2 + \ldots + m_l + p \leqslant K - 1\}$$

is finite dimensional and a finite dimensional representation $T^{(\mu,k)}$ of D can be induced in this quotient space. Its dimension is

$$D(K, l)=\sum_{i=1}^{K-1} \frac{(l+i)!}{l!i!}. \tag{3.3}$$

To write the above finite dimensional representation explicitly, we define

$$f_{K,\mu}(M)=\theta\left(N-1-\sum_{i=1}^{l} m_i-p\right)|m_j, p\rangle \bmod V(K)$$

for

$$M=(m_1, m_2, \ldots m_l, p) \qquad m_j, p\in Z^+, j=1, 2, \ldots l$$

and

$$\theta(x)=\begin{cases}1 & \text{if } x\geqslant 0\\ 0 & \text{if } x<0\end{cases}$$

On this basis, we have an explicit expression for the representation $T^{(\mu,k)}$:

$$\begin{aligned}
\bar{a}_i f_{K,\mu}(M)&=f_{K,\mu}(M+e_i)\\
a_i f_{K,\mu}(M)&=m_i f_{K,\mu}(M-e_i+e_{l+1})\\
E f_{K,\mu}(M)&=f_{K,\mu}(M+e_{l+1})\\
\bar{N} f_{K,\mu}(M)&=\left(\sum_{i=1}^{l} m_i+\mu\right) f_{K,\mu}(M)
\end{aligned} \tag{3.4}$$

where

$$\begin{aligned}
e_1&=(1, 0, 0, \ldots, 0, 0)\\
e_2&=(0, 1, 0, \ldots, 0, 0)\\
&\vdots\\
e_l&=(0, 0, 0, \ldots, 1, 0)\\
e_{l+1}&=(0, 0, 0, \ldots, 0, 1)
\end{aligned}$$

are the unit vectors in the lattice space

$$Z^{l+1}: \{M=(m_1, m_2, \ldots m_l, p)\,|\, m_j, p\in Z^+, j=1, 2, \ldots l\}.$$

Note that the representatives of N^k can be given through (3.3) and

$$N=\ln(1-E)=-\sum_{s=1}^{\infty} \frac{E^s}{s}$$

$$N^k=\sum_{s=0}^{\infty} C(K)_s E^s$$

where $C(K)_s$ can be explicitly determined.

4. An example of a new R-matrix for the quantum double D

Using the above obtained finite dimensional representation $T^{(\mu,K)}$ defined by (3.2), the new R-matrix can be constructed as

$$\begin{aligned}R(H_1, H_2) &\doteq R(\mu_1, \mu_2; K_1, K_2)\\ &= T^{(\mu_1,K_1)}\otimes T^{(\mu_2,K_2)}(\hat{R})\in \mathrm{End}[Q(K_1, \mu_1)\otimes Q(K_2, \mu_2)].\end{aligned}$$

The general construction of quantum double theory maintains that the above R-matrix satisfies the QYBE. Here, the extra parameters μ appear as the colour parameters [20] which, in general, are different from the non-additive dynamic spectrum parameters such as in the exactly-solvable models in statistical mechanics [33–41]. Let

$$f_i(M_i) = f_{K_i,\mu_i}(M_i) \qquad i = 1, 2.$$

We can calculate the elements for $R(H_1, H_2)$ in terms of the following actions

$$\begin{aligned}R(H_1, H_2)f_1(M_1)\otimes f_2(M_2) &= \sum_{M'_1,M'_2} R(H_1, H_2)^{M'_1,M'_2}_{M_1,M_2} f_1(M'_1)\otimes f_2(M'_2)\\ &= \sum_{j,k,n_i}^{j_{\max},k_{\max},n_{i,\max}} \frac{C(K)_s}{k!}\left(\sum_{i=1}^{l}[m_i+n_i]+\mu_1\right)\\ &\quad\times\prod_{i=1}^{l}\frac{m_i!}{(n_i!)^2} f_1\left(M_1+\sum_{i=1}^{l} n_i e_i\right)\otimes f_2\left(M_2-\sum_{i=1}^{l} n_i e_i\right.\\ &\quad\left.+\left(\sum_{i=1}^{l} n_i+k\right)e_{l+1}\right)\end{aligned}$$

where $x_{\max}$ for $x = j, k, n_j, j = 1, 2, \ldots, l$ are the maximum values that x can take so that all the terms in the above expression make sense. Now, we consider an extremely special example with $K_1 = K_2 = 2 = l$, $\mu_1 = \mu_2 = \mu$. In this case, the corresponding infinite dimensional representation of D

$$\begin{aligned}\bar{a}_1|m.n, p\rangle &= |m+1, n, p\rangle\\ \bar{a}_2|m, n, P\rangle &= |m.n+1, p\rangle\\ a_1|m, n, p\rangle &= m|m-1, n, p+1\rangle\\ a_2|m, n, p\rangle &= n|m, n-1, p+1\rangle\\ E|m, n, p\rangle &= |m, n, p+1\rangle\\ \bar{N}|m, n, p\rangle &= (m+n+\mu)|m, n, p\rangle\end{aligned} \tag{4.1}$$

induces a four-dimensional representation on $Q(K=2, \mu)$

$$\begin{aligned}&\bar{a}_1 = E(2, 1) \qquad \bar{a}_2 = E(3, 1) \qquad E = E(4, 1)\\ &\bar{N} = \mu E(1, 1) + (1+\mu)(E(2, 2)+E(3, 3)) + \mu E(4, 4)\\ &a_1 = E(4, 2) \qquad a_2 = E(4, 3) \qquad N = E(4, 1)\end{aligned} \tag{4.2}$$

where $E(i,j)$ is 4×4 matrix unit such that $E(i,j)_{m,n}=\delta_{im}\delta_{jn}$;

$$|m, p, n\rangle = |, m_1=m, m_2=n, p\rangle.$$

Through the universal R-matrix, this explicit representation of D results in a 16×16 R-matrix

$$R=\begin{bmatrix} I+\mu N & 0 & 0 & 0 \\ a_1(I+\mu N) & I+(1+\mu)N & 0 & 0 \\ a_2(I+\mu N) & 0 & I+(1+\mu) & 0 \\ 0 & 0 & 0 & I+\mu N \end{bmatrix}$$

where μ is a complex parameter and I a 4×4 unit matrix.

5. Remarks

1. The usual quantum doubles are the q-deformations of the universal algebras and possess a 'standard' quantum double structure such that both the subalgebras A and B are non-commutative and non-co-commutative. This symmetric structure reflects the duality of A and B. Note that these standard quantum doubles approach the usual universal enveloping algebras (UEA) in the classical limit $q\to 1$. In this paper, we have constructed so-called non-standard quantum doubles that are not those q-deformations and possess asymmetric dual structure such that one of the subalgebras A and B is commutative but non-co-commutative and another co-commutative but non-commutative. As new quasi-triangular Hopf algebras, these QDs naturally enjoy the QYBE, but they have not the usual classical limit.

2. Up to now, we have established new quasi-triangular Hopf algebra as the quantum double D of A and B, which is defined by (2.1), (2.5), (2.6) and proposition 1. Now, we show how the l-state boson algebra can be embedded into the quantum double D as a Hopf subalgebra. To this end, we use $E=1-e^N$ and assume $E-1$ is invertible. Then,

$$[a_i, \bar{a}_j]=\delta_{i,j}E \qquad [E, \text{everything}]=0 \tag{5.1}$$

that is to say, the operators $\bar{a}_i$, a_i and $\bar{N}$ behave as creation operators, annihilation operators and the number operator respectively. They just generate the l-state boson algebra. The structure of the quantum double D naturally induces for the l-state boson algebra a Hopf algebraic construction

$$\begin{aligned}
&\Delta(a_i)=a_i\otimes(1-E)+1\otimes a_i \\
&\Delta(E)=E\otimes E-E\otimes 1-1\otimes E \\
&S(E)=-\frac{E}{E-1} \qquad S(a_i)=-\frac{a_i}{1-E},\ S(1)=1 \\
&\varepsilon(a_i)=0=\varepsilon(E) \qquad \varepsilon(1)=1.
\end{aligned} \tag{5.2}$$

The above discussion shows that the usual boson algebra enjoys the quantum group structure as well as the q-deformed boson that realized quantum algebras [46–48].

3. In a recent paper [49], from the simplest non-co-commutative Hopf algebra with two commuting generators, we built a quasi-triangular Hopf algebra H as its quantum double which is non-commutative. As an associative algebra, it is similar to the result obtained in [50] by taking a limit of a q-deformation of $SU(2)$. It is easy to observe that this algebra H is just an isomorphism to the special case with $l=1$ of the general non-standard quantum double in this paper. The method in [46] can be regarded as an inverse process of that in this paper in a special case. This means that the inverse construction of the present study possibly shows a 'quantization' from commutative objects to certain non-commutative ones.

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